Interaction of a Laminar Hypersonic Boundary Layer and a Corner Expansion Wave

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There are three distinct effects involved in the interaction of a laminar boundary layer and a supersonic corner expansion wave. These are: the upstream influence effect causing some pressure decay ahead of the corner, transverse pressure gradients in the immediate neighborhood of the corner, and the interaction of the boundary layer downstream with the external flow. Arguments are presented to suggest that, when the flow is locally hypersonic and the wall is highly cooled, the dominant effect is the downstream interaction process. Hence the major features can be calculated by using a simple interaction analysis downstream of the corner based on the Prandtl boundary-layer equations and the equations of an inviscid noncentered simple wave. Numerical results are obtained by using the "cold wall" similarity solution to the boundary-layer equations. These show that pressure decay extends over a region which can be many times larger than the original plate length used to generate the boundary layer.

Nomenclature

$C_{f\infty}$ = skin-friction coefficient; $C_{f\infty} = (2/\rho_\infty U^2)(\mu\partial u/\partial y)|\mu\partial u/\partial y|_w$
C_∞ = Chapman-Rubesin factor, $\mu_w T_\infty/\mu_\infty T_w$
f = $\int(u/u_e)d\eta$, defined by Eq. (3.6)
g = H/H_e
G = $\int_0^\infty \{g - (f')^2\}d\eta$, see Eq. (3.14)
h = static enthalpy, $h = [\gamma/(\gamma - 1)]\ p/\rho$ for a perfect gas
H = total enthalpy, $H = h + \frac{1}{2}(u^2 + v^2) \simeq h + \frac{1}{2}u^2$
k = thermal conductivity
K = $M_\infty d\delta^*/dx$
M = Mach number
N = $\rho\mu/\rho_w\mu_w$
p = static pressure
P = p_e/p_∞
R = $\int P\, dZ$
Re = Reynolds number; $Re_{x,\infty}$ is based on running length x and freestream properties
St_∞ = Stanton number; $St_\infty = q_w/\rho_\infty U C_p(T_r - T_w)$
T = static temperature
u,v = components of velocity in the x and y directions, respectively
U = freestream velocity
x,y = co-ordinates along the wall and normal to it, respectively
Z = $\rho_\infty U M_\infty^{-6} x/\mu_\infty\ C_\infty = 1/\bar{\chi}_\infty^2$
α_c = amount by which the flow is turned in the region where the boundary-layer equations do not apply
α_t = $\alpha_w + d\delta^*/dx|_c$; total effective turning angle at the corner
α_w = corner turning angle of body
γ = C_p/C_v; specific heat ratio
Γ = $(\int P dZ)^{1/2}/P$
δ = boundary-layer disturbance thickness
δ^* = boundary-layer displacement thickness

$$\delta^* = \int_0^\delta \left(1 - \frac{\rho u}{\rho_e u_e}\right)dy$$

η,ξ = defined by Eq. (3.5)
μ = viscosity
ρ = density
$\bar{\chi}_\infty$ = viscous interaction parameter, $\bar{\chi}_\infty = M_\infty^3\left(\frac{C_\infty}{Re_{x,\infty}}\right)^{1/2}$

Subscripts

c = conditions immediately upstream of the corner calculated as if the corner was not present
d = conditions immediately downstream of the corner
e = conditions at the edge of the boundary layer
o = isentropic reservoir or stagnation conditions
w = conditions at the wall
∞ = conditions in the freestream

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1. Introduction

THE interaction of a laminar supersonic or hypersonic boundary layer with a steady corner expansion wave is a problem of considerable current theoretical and practical interest. When the boundary layer remains attached downstream of the corner, the turning process involves at least three mechanisms. First, the presence of the corner is signalled upstream through the subsonic portion of the boundary layer, and the wall pressure commences to fall ahead of the corner. Second, in the immediate neighborhood of the corner, the boundary-layer-created shear flow is turned by predominantly inviscid forces. Pressure gradients normal to the streamlines should be significant, and viscous shear stresses are believed to be important only in a very small layer near the wall, where the zero slip condition must be satisfied. Finally, continuity of pressure along the wall and the pressure gradient required to sustain centrifugal acceleration require that immediately downstream of the corner, the turning process in the external inviscid flow, which is assumed to be accomplished through a noncentered simple wave, is incomplete. Therefore in the region downstream of the corner an interaction occurs between the inviscid flow and boundary layer. The net result is an asymptotic decay in surface pressure to the Prandtl-Meyer value for the wall turning angle α_w.

For moderate and high Mach numbers the downstream interaction process can extend many boundary-layer thicknesses beyond the corner. The relative importance of the three mechanisms depends on α_w, the Mach number M_c of the inviscid flow just upstream of the corner, and the wall temperature T_w. In addition, separation of the boundary layer can occur at or near the corner for certain downstream geometries and the resultant flow is then very complex. The present work is concerned only with the attached flow case.

Attempts have been made to treat the problem within the framework of boundary-layer theory.[1-6] Zakkay et al.[1,2]

calculated heat-transfer rates for cone cylinders downstream of the corner by treating the flow upstream of the corner as if the corner was not present. They allowed the boundary-layer-created shear flow to undergo frictionless expansion around the corner, and then evaluated the subsequent development of the boundary layer by representing it as a new boundary layer which starts at the corner, on top of which the upstream-developed boundary layer turned by the corner was matched as a viscous shear layer. The pressure distribution downstream of the corner was an input to the calculations determined from an inviscid solution for cone cylinder geometries. The displacement effects of the downstream boundary layer on the external inviscid flow were not considered in the formal analysis, which used a Gortler type series expansion of the boundary-layer equations.

Hunt and Sibulkin[3] examined the change in momentum thickness and shape factor δ^*/Θ of the boundary layer through the corner region by means of a momentum integral technique modified to account for radial pressure gradients. They predicted large changes in the momentum thickness of the boundary layer as it passed through the corner region. Weinbaum[4] examined the flowfield in the immediate neighborhood of the corner by using the method of characteristics in the supersonic portion of the flow and assuming that the sonic line remained parallel to the wall downstream of the corner. His results suggested that for locally hypersonic flows and small turning angles, the flow was highly under-expanded just downstream of the corner.

More recently, Oosthuizen[5] undertook an extensive analysis of the problem. His work, which was a generalization of an earlier and more simplified treatment by Curle,[6] used the momentum integral form of the boundary-layer equations and the equations of a noncentered Prandtl-Meyer simple wave to describe the flow. He calculated the flowfield both upstream and downstream of the corner region by allowing the boundary layer and expansion wave to interact, and matched the upstream and downstream solutions by specifying continuity of inviscid flow properties and by matching the shape factor δ^*/Θ of the boundary layer at the corner. His analysis ignored centrifugal effects in the neighborhood of the corner and was limited to moderately supersonic M_c.

It must be noted that the use of the boundary-layer equations to model upstream influence effects in supersonic flow involves conceptual difficulties. It has been pointed out[7] that, since the boundary-layer equations are parabolic and the inviscid supersonic flow equations are hyperbolic, the possibility of disturbances traveling upstream are precluded. Hence the elliptic behavior is required to correctly model upstream influence. Since the mechanism is propagation of signals in the subsonic portion of the boundary layer, the equations for this portion should be elliptic. This type of approach has been used by several authors, perhaps the most complete of which is the paper by Lighthill.[8] Lighthill treats the boundary layer as a parallel shear flow, on top of which a perturbation was superimposed which was inviscid in the subsonic portion of the boundary layer with the exception of the region immediately adjacent to the wall. His analysis is limited to small disturbances and moderate M_e. However, it has been pointed out[9] that there is often spurious changes of type in approximation schemes for partial differential equations. Consequently undue significance should not be attached to the failure of the boundary-layer equations to correctly represent upstream influence, so long as they adequately represent the conservation laws. Hence the use of Prandtl's equations in the manner of Refs. 5 and 6 is acceptable provided significant upstream influence extends far beyond the neighbourhood of the corner where transverse pressure gradients are important in the boundary layer.

In the following section it is argued that for locally hypersonic flows and highly cooled walls the upstream influence effect is confined to a relatively small region near the corner, where the boundary-layer equations are inapplicable. Further, because there are large pressure changes associated with expanding hypersonic flows, the major feature is the downstream interaction process and a greatly extended region of pressure decay.

2. Behavior in Hypersonic Flow

A characteristic feature of inviscid hypersonic flow is that small angle expansion turns generate large decreases in pressure and density, but very small changes in fluid speed. Typically, an expansion of a perfect gas with a specific heat ratio $\gamma = 1.4$ through a 10° turning angle by a simple wave from $M_e = 10$ causes the pressure to drop by a factor of 21 and the density by a factor of 9, whereas the speed increases by less than 2%. This affects the growth of a limiting hypersonic boundary layer. A decrease in pressure at the edge of the boundary layer is associated with a negligible velocity change so that the principal effect on the boundary layer is a decrease in the density and an increase in the displacement thickness. It is in direct contrast with the behavior of the incompressible boundary layer, where a decrease in pressure tends to thin the boundary layer because the only effect is an increase in the velocity at the edge of it. For supersonic boundary layers if M_e is low enough, it is possible for the boundary layer to thin, or at least grow at a slower rate than it does under constant pressure.

This is a particular example of the behavior first reported in the literature by Crocco and Lees[10] in connection with their study of the supersonic base flow problem. They introduced the concept of a critical Mach number $M_{ecr} > 1$ for a given isentropic external flow. For $M_e < M_{ecr}$, $d\delta^*/dp_e > 0$, whereas for $M_e > M_{ecr}$, $d\delta^*/dp_e < 0$. The two modes of behavior have been called subcritical and supercritical, respectively. A supercritical boundary layer has a relatively small subsonic region and acts as if it is a supersonic streamtube. The value of M_e at which the boundary layer goes supercritical is obviously dependent on T_w, since this affects the location of the sonic line; but it may be safely assumed that hypersonic boundary layers are supercritical if the wall is highly cooled.[11]

The significance of the critical point is that, within the framework of boundary-layer theory, for $M_e < M_{ecr}$ the boundary layer will thin under the action of an isentropic simple wave expansion generated by the displacement effects of the boundary layer. For $M_e > M_{ecr}$ the boundary layer will thicken, and a contradiction arises since this will tend to generate a compression wave. Consequently, the type of analysis used by Oosthuizen[4] in which upstream of the corner, the boundary-layer equations were matched to an expansion wave is only possible if $M_e < M_{ecr}$. In the compression corner problem, a corresponding contradiction occurs, and it has been suggested that the boundary layer undergoes a supercritical-subcritical jump[11] just before the interaction process begins. This jump is expected to occur in a region which is a few boundary-layer thicknesses in extent, and in which the sonic line moves rapidly away from the wall. It is difficult to conceive of the occurrence of such a jump in the expansion corner problem.

Clearly, upstream influence effects scale with the subsonic layer thickness at the corner, and in a supercritical boundary layer this should be small relative to δ_c. Hence, it is to be expected that for sufficiently large M_e, significant upstream influence effects are confined to a region which is not large relative to δ_c and in which centrifugal effects are important, so that the Prandtl boundary-layer equations cannot be used. In the locally hypersonic flow, it is necessary to use a more realistic analysis such as that given by Lighthill[8] or more recently by Olsson and Messiter[12] and Weiss and Nelson[13] for the base flow problem.

The implication of the preceding remarks is that no mechanism exists by which large pressure decay can occur upstream of the corner. Since in the hypersonic flow even small values of

α_w can be associated with very large changes in pressure through a homentropic simple wave, continuity of pressure along the wall requires that almost all of the pressure decay occur downstream of the corner. Further, because the velocity at the edge of the boundary-layer remains practically constant during the expansion process, within the boundary layer the density decrease in a given streamtube must be primarily associated with increase in streamtube thickness. Hence, large increases in δ^* are expected. On the other hand, the boundary, layer thickness cannot grow too rapidly, since it has to allow the external inviscid flow to turn and expand. It is to be noted that the density changes that occur within the boundary-layer as it grows downstream of the corner will be controlled by the pressure changes at the edge, rather than the density changes at the edge. The reason for this is that the temperature distribution in the bulk of a locally hypersonic boundary layer is controlled by the wall temperature and the stagnation temperature of the inviscid flow, and is practically independent of the static temperature at the edge of the boundary layer. This is the Mach independence principle for hypersonic boundary layers.[14] Since, in an inviscid expansion the pressure decrease is usually much greater than the density decrease, the extent to which the boundary layer has to grow to accommodate the pressure decrease is greater than consideration of density effects alone will indicate.

These effects suggest strongly that for small turning angles α_w, the pressure decay process caused by the interaction of the inviscid flow and the boundary layer will be spread out, and further that the great majority of the pressure decay downstream of the corner should occur in a region where centrifugal effects in the boundary layer are not important so that Prandtl's equations can be applied. This behavior is illustrated in Fig. 1. If the external inviscid flow is turned through an angle α_c in the corner region where Prandtl's equations do not apply, then it is assumed that $\alpha_c/\alpha_w \ll 1$. Calculations of the extent of the interaction process based on this simplified model are presented in this paper.

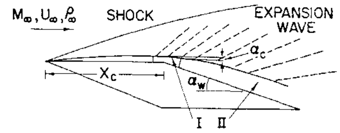

Fig. 1 Proposed model for hypersonic flow and small turning angles α_w. Region I is the area where the boundary-layer equations are presumed to be inapplicable, and region II is the downstream interaction region.

3. Cold Wall Similarity in Hypersonic Boundary Layers

A number of methods for the calculation of laminar hypersonic boundary layers for arbitrary external pressure distributions are described in the literature.[14] Integral methods[11,15] for example, have been developed to apply to this type of interaction problem. However, in view of the assumptions used in the present model, the "cold wall similarity" method suggested by Lees[16] was applied, since it enabled a very simple formulation to be developed. In spite of its simplicity, for favorable pressure gradients and with $T_w/T_o \ll 1$ this method is known to yield reasonably accurate results. Cold wall similarity theory is developed below in a form suitable for the present calculations.

For two-dimensional bodies the boundary-layer equations are

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{3.1}$$

$$\rho u(\partial u/\partial x) + \rho v(\partial u/\partial y) = -(dp_e/dx) + (\partial/\partial y)[\mu(\partial u/\partial y)] \tag{3.2}$$

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial H}{\partial y}\right) + \frac{\partial}{\partial y}\left[\mu\left(1 - \frac{1}{Pr}\right)\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right)\right] \tag{3.3}$$

where, since in the boundary layer $v \ll u$

$$H = h(p,\rho) + \tfrac{1}{2}u^2 \tag{3.4}$$

The usual transformations of compressible boundary-layer theory are applied[14]

$$\xi = \int_0 \rho_w \mu_w u_e dx; \quad \eta = \frac{u_e}{(2\xi)^{1/2}} \int \rho dy \tag{3.5}$$

$$u/u_e = \partial f/\partial \eta; \quad g = H/H_e \tag{3.6}$$

Continuity is automatically satisfied, so that if the solutions are assumed to be a function of η only the equations reduce to

$$(Nf'')' + ff'' + \frac{2}{u_e}\frac{du_e}{d\xi}\left\{\frac{\rho_e}{\rho} - f'^2\right\} = 0 \tag{3.7}$$

$$\left(\frac{N}{Pr}g'\right)' + fg' + \frac{u_e^2}{H_e}(\xi)\left\{N\left(1 - \frac{1}{Pr}\right)f'f''\right\}' = 0 \tag{3.8}$$

where $\partial/\partial\eta = (\ \ )'$ and $N = \rho\mu/\rho_w\mu_w$.

The boundary conditions are, with u_e (ξ) or p_e (ξ) given

$$\text{at } y \text{ or } \eta = 0, f = f' = 0, g = g_w(\xi)$$
$$\text{as } \eta \to \infty, f' \to 1, g \to 1 \tag{3.9}$$

In general, ξ-independence, or self similarity exists only under very restricted circumstances. If it is assumed that 1) the flow is locally hypersonic, so that $u_e^2 = 2H_e = U^2$, 2) the gas is calorically perfect so that $h = \gamma/(\gamma - 1)\ p/\rho$; $p/\rho = RT$, 3) Pr = constant, 4) $\mu \propto T$ so that $N = 1$, and 5) g_w = constant, Eqs. (3.7) and (3.8) reduce to

$$f''' + ff'' + \beta(x)(g - f'^2) = 0 \tag{3.10}$$

$$g'' + Prfg' + 2(Pr - 1)(f'f'')' = 0 \tag{3.11}$$

where $\beta(x) = \gamma - 1/\gamma \int p_e dx/p_e^2\ dp_e/dx$.

Self similarity exists only if β = constant or $p_e \propto x$. If, in place of assumption (5), it is assumed that $g_w \ll 1$, then the boundary conditions suggest that the term $\beta(x)\ (g - f'^2) \ll 1$ over the range of integration. The boundary equations are then approximately self similar for arbitrary $p_e = p_e(x)$, since the momentum becomes approximately

$$f''' + ff'' = 0 \tag{3.12}$$

The "cold wall" similarity approach is distinct from the local similarity method[17] which treats β as a parameter which varies slowly with x along the boundary layer. The local values of dp_e/dx and $\int p_e dx$ are used to determine β and the boundary-layer profiles are then determined from the similar solutions for the same value of β.

For self similar profiles the displacement thickness δ^* is given by

$$\delta^* = \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy = \frac{(2\xi)^{1/2}}{\rho_e u_e}\int_0^\infty \left(\frac{\rho_e}{\rho} - \frac{u}{u_e}\right) dy \tag{3.13}$$

Application of the assumptions 1 to 5 mentioned before leads to

$$\delta^* = \frac{\gamma - 1}{\gamma}\, U^{3/2}\left(\frac{2\mu_w}{RT_w}\right)^{1/2} \frac{[\int_0 p_e dx]^{1/2}}{p_e(x)}\, G(Pr) \tag{3.14}$$

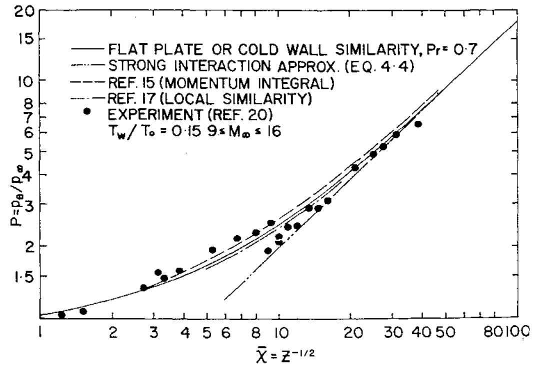

Fig. 2 Comparison of various approximate methods for predicting the pressure distribution on a flat plate in laminar hypersonic flow with experimental results.

where

$$G \equiv \int_0^\infty [g - (f')^2]d\eta \tag{3.15}$$

It is convenient to write this relation in terms of the appropriate freestream variables M_∞, p_∞, and μ_∞. After some algebra it turns out that (3.14) can be cast into the form

$$M_\infty \delta^*/x = (\gamma - 1)/(2)^{1/2} G(Pr)/Z[\Gamma(Z)] \tag{3.16}$$

where

$$P = \frac{p_e}{p_\infty}\, Z = \frac{R_{ex,\infty}}{M_\infty{}^6 C_\infty},\ R_{ex,\infty} = \frac{xU\rho_\infty}{\mu_\infty},\ \Gamma = \frac{(\int PdZ)^{1/2}}{P} \tag{3.17}$$

The quantity Z is proportional to x and is related to the hypersonic viscous interaction parameter by $\bar{\chi}_\infty = Z^{-1/2}$.

Simple expressions for heat transfer and skin friction can be similarly derived

$$St_\infty M_\infty{}^3 = [g_w'/(2)^{1/2} Pr\{g_r - g_w\}]\Gamma^{-1} \tag{3.18}$$

$$C_{f\infty} M_\infty{}^3 = (2)^{1/2} f''(o)\Gamma^{-1} \tag{3.19}$$

where $g_r = H_r/H_e$ and H_r is the recovery enthalpy.

It has been pointed out[18] that, even if the cold wall assumption does not apply and the pressure gradient term in Eq. (3.7) is retained, its contribution can be small for favorable pressure gradients. From Eq. (3.2), for β = constant

$$p_e \propto x^n;\ \beta = [(\gamma - 1)/\gamma][n/(n + 1)]$$

so that if $p_e \propto x^{-1/2}$, as in strong shock boundary-layer interaction,[14] when $\alpha = 1.40$, $\beta = 0.286$. Hence, the cold wall similarity concept should be applicable over larger values of g_w than is suggested solely by consideration of the term $[g - (f')^2]$.

Since in the present analysis the momentum equation is reduced to Blasius' equation, then $f_w'' = 0.470$. To obtain values of G and g_w, the solution of (3.11) is available in closed form and is $g(\eta) = g_w + f'(\eta)\,(1 - g_w)$, so that

$$G = g_w \int_0^\infty (1 - f')d\eta + \int_0^\infty f'(1 - f')d\eta$$

Since the two integrals are obtained directly from the solution of the Blasius equation, then $G = 0.4696 + 1.2168\, g_w$. Solutions for other values of Pr have been quoted by Dewey[17] and by Sullivan.[19]

4. Application to Shock Boundary-Layer Interaction Theory

The growth of the boundary layer on a flat plate is now calculated by the present theory. This is necessary to provide the initial condition, that is P and $R = \int PdZ$, for the expansion wave boundary-layer interaction process downstream of the corner. It also serves as a convenient check on the accuracy of the present theory since a direct comparison with more accurate calculations of this problem can be made.

For shock boundary-layer interaction calculations the tangent wedge rule is normally used to estimate the pressure at the edge of the boundary layer.[14] In this problem the effective body is usually slender, so that the pressure at the edge of the boundary layer is accurately given by the hypersonic small disturbance solution for oblique shocks. The tangent wedge relation is[14]

$$P - 1 = \gamma K^2\left[\left\{\left(\frac{\gamma + 1}{4}\right)^2 + \frac{1}{K^2}\right\}^{1/2} + \frac{\gamma + 1}{4}\right] \tag{4.1}$$

where $K = M_\infty d\delta^*/dx$.

Now differentiate Eq. (3.16) and insert into Eq. (4.1) to obtain the following expression for the shock boundary-layer interaction problem:

$$\frac{dP}{dZ} = \frac{P^2}{2R}\left[1 - \left\{\frac{4\,(P - 1)}{\gamma(\gamma - 1)G(Pr)}\right\} \times \left\{\frac{\gamma R}{P(\gamma + 1) + (\gamma - 1)}\right\}^{1/2}\right] \tag{4.2}$$

where $dR/dZ = P$.

Solution of the two ordinary differential equations by standard Runge Kutta techniques leads to a pressure distribution $P = P\ (Z)$ or $p_e = p_e(x)$. It is known in the strong interaction limit $\bar{\chi}_\infty \rightarrow \infty$ or $Z \rightarrow 0$, that $P \rightarrow \infty$. Hence, to start the integration the appropriate expression corresponding to the strong interaction limit must be provided. This is done by simplifying (5.3) with the approximation $P \gg 1$ to obtain

$$\frac{2R}{P^2}\frac{dP}{dZ} = 1 - \frac{4}{(\gamma - 1)G}\left[\frac{PR}{\gamma(\gamma + 1)}\right]^{1/2} \tag{4.3}$$

Note that as $P \rightarrow \infty$, $R \rightarrow 0$ so that both terms in the square brackets of Eq. (5.3) have to be retained. By assuming a solution of the form $P = AZ^n$ it is found that

$$P = \tfrac{3}{8}(\gamma - 1)G[2\gamma(\gamma + 1)/Z]^{1/2} \tag{4.4}$$

This solution agrees with that given in Ref. 14, pp. 358–9. Integration of Eq. (4.2) then proceeds by obtaining starting values of P and R at a value of $\bar{\chi}_\infty = Z^{-1/2}$ which is chosen such that the strong interaction solution (4.4) and the complete solution give the same value of dP/dZ to within acceptable error.

The heat transfer and pressure distribution on a cold flat plate in hypersonic flow were computed by the cold wall similarity method and the results are given in Figs. 2 and 3.

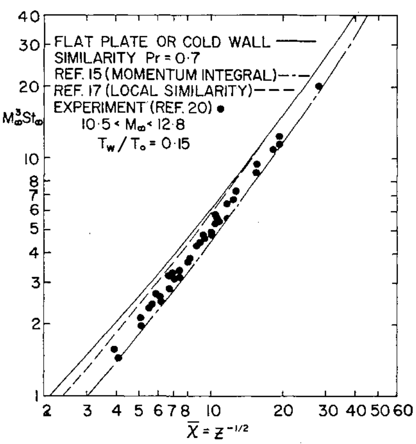

Fig. 3 Comparison of various approximate theories for heat transfer to a flat plate in hypersonic flow with experiment.

The validity of the present approximation was verified by comparison with other theoretical methods and some experimental results. The theoretical methods are the "local similarity" approach described in detail by Dewey[17] and a momentum integral method described by Chan,[15] and the experimental results were obtained by Hall and Golian.[20] The strong interaction solution as given by Hayes and Probstein[14] is included in the comparisons of pressure distributions. Excellent agreement is obtained between the theories and experiment in the case of the pressure distribution. Agreement in the case of the heat-transfer distribution is somewhat less satisfactory; but the "flat plate" similarity method used here does not appear to be significantly worse than the other methods.

5. Formulation of the Downstream Flowfield

The rate of growth of the boundary-layer displacement thickness immediately downstream of the corner region $d\delta^*/dx|_d$ is given by (see Fig. 1)

$$d\delta^*/dx|_d = d\delta^*/dx|_c \pm \alpha_w - \alpha_c \tag{5.1}$$

where $d\delta^*/dx|_c$ is the rate of growth of the boundary-layer displacement thickness immediately upstream of the corner. For the present analysis $d\delta^*/dx|_c$ is assumed to be that value which would occur if no corner were present at $x = x_c$. Also with $\alpha_c \ll \alpha_w$, Eq. (5.1) can be written

$$d\delta^*/dx|_d \simeq d\delta^*/dx|_c + \alpha_w = \alpha_t \tag{5.2}$$

That is to say the turning process is, in the first approximation, assumed to be carried out entirely in the region where the boundary-layer equations apply and therefore continuity of pressure $p_c = p_d$.

Since the inviscid flow is assumed to be a simple wave, the pressure downstream of the corner at the edge of the boundary layer is given by[21]

$$\frac{P}{P_c} = \left\{1 - \frac{\gamma - 1}{2}\left[\alpha_t - \frac{d\delta^*}{dx}\right]M_c\right\}^{2\gamma/\gamma-1} \tag{5.3}$$

where the hypersonic small disturbance approximations have been used. Equation (5.3) can be combined with Eq. (3.16) to give the governing equation for the present problem

$$\frac{dP}{dZ} = \frac{P^2}{2R}\left(1 - \frac{2(2R)^{1/2}}{(\gamma - 1)G}\left[M_\infty\alpha_T + \left(\frac{M_\infty}{M_c}\right)\frac{2}{\gamma - 1}\left\{\left(\frac{P}{P_c}\right)^{\gamma-1/2\gamma} - 1\right\}\right]\right) \tag{5.4}$$

The initial conditions are obtained from the shock boundary-layer interaction solution upstream of the corner

$$\text{at } Z = Z_c,\ P = P_c,\ R = R_c \tag{5.5}$$

which in turn are obtained from the solution of Eq. (4.2).

The two quantities $M_\infty\alpha_t$ and M_c/M_∞ have to be specified to complete the formulation. Since the tangent wedge formula was used to relate the flow deflection to the pressure, $d\delta^*/dx|_c$ can be written in terms of P

$$M_\infty\left.\frac{d\delta^*}{dx}\right|_c = \frac{P_c - 1}{\gamma}\left[\frac{2\gamma}{P_c(\gamma + 1) + (\gamma - 1)}\right]^{1/2} \tag{5.6}$$

Hence

$$M_\infty\alpha_t = M_\infty\alpha_w + \frac{P_c - 1}{\gamma}\left[\frac{2\gamma}{P_c(\gamma + 1) + (\gamma - 1)}\right]^{1/2} \tag{5.7}$$

The choice of a suitable value of M_c is not quite as straightforward. The use of the tangent wedge concept to compute M_c is in general inadequate, since the fluid streamlines at the edge of the boundary layer at a given value of x or Z cross the shock wave at point where the entropy increase can be much higher than would be computed by the tangent wedge formula. Consequently the sound speed estimated by using the tangent wedge formula should be low. However, it can be argued that it is not reasonable to apply a simple wave description to the inviscid flow if there is a large difference between the Mach numbers at the edge of the boundary layer and at a point just behind the shock wave at the same value of x_c. Hence, it is necessary to restrict the present model to values of x or Z sufficiently large that x_c is not in the strong interaction regime where $\bar{\chi}_\infty \gg 1$. It is then consistent to use the oblique shock wave relations to estimate M_c. The required relation is obtained from hypersonic small disturbance theory and is[21]

$$M_c^2 = \left\{\frac{(\gamma + 1)P_c + (\gamma - 1)}{(\gamma - 1)P_c + (\gamma + 1)}\right\}\frac{1}{P_c} \tag{5.8}$$

An additional reason for the constraint that $\bar{\chi}_\infty$ should not be too large at the corner is that in the strong interaction regime the shock wave is relatively close to the body. Then reflections of the corner expansion wave from the shock could intersect the boundary layer relatively close to the corner, and cause the pressure distribution to deviate significantly from the simple wave law used here.

The present theory is readily generalized to the case when the upstream body is a wedge. For a wedge angle θ followed by a corner of angle α_w the flowfield upstream of the corner is given by

$$\frac{dP}{dZ} = \frac{P^2}{2R}\left[1 + \frac{2(2R)^{1/2}}{G(\gamma - 1)}\left\{K_\theta - \frac{(P - 1)}{\gamma} \times \left(\frac{2\gamma}{P(\gamma + 1) + (\gamma - 1)}\right)^{1/2}\right\}\right] \tag{5.9}$$

where $K_\theta = M_\infty\theta$. The equation for the flow downstream of the corner is the same as for the flat plate model. However, the expression for α_t is different because P_c now gives the quantity $M_\infty[\theta + d\delta^*/dx|_c]$; so that

$$M_\infty\left.\frac{d\delta^*}{dx}\right|_c = \frac{P_c - 1}{\gamma}\left[\frac{2\gamma}{P_c(\gamma + 1) + (\gamma - 1)}\right]^{-1/2} - M_\infty\theta \tag{5.10}$$

Then α_t is given by Eq. (5.2) as before. The expression for M_c remains the same as for the flat plate case.

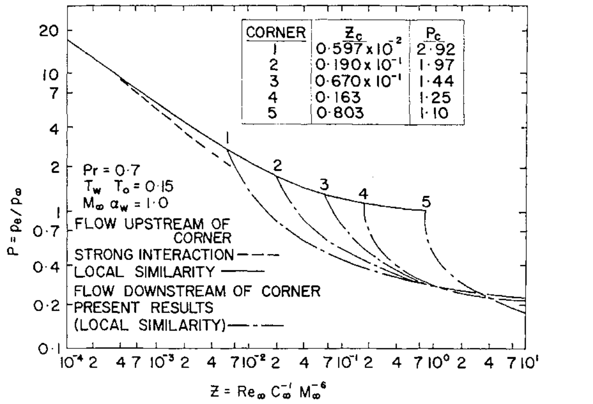

Fig. 4 Pressure distribution downstream of an expansion corner for $M_\infty\alpha_w = 1.0$ and $T_w/T_0 = 0.15$.

6. Results and Discussion

Some calculated values of surface pressure are given in Fig. 4. The forebody is a flat plate. Inspection of the governing equations shows that the solution downstream of the corner depends on M_c and α_w only through the parameters P_c and $M_\infty\,\alpha_w$. The results given in Fig. 4 are for $Pr = 0.70$,

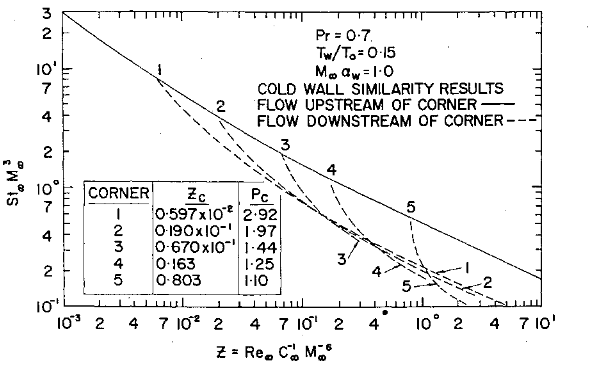

Fig. 5 Heat-transfer distribution downstream of an expansion corner in hypersonic flow for $M_\infty \alpha_w = 1.0$ and $T_w/T_0 = 0.15$.

$g_w = 0.15$, and $M_\infty\ \alpha_w = 1.0$. The most striking feature of these results is the greatly extended region of pressure decay, this can occur over a region which can be much longer than x_c. Typically, at a value of $Z_c = 0.0190$ or $\bar{\chi}_\infty = 7.25$, $P_c = 1.97$, and the pressure downstream of a corner at this point decays to 49% of P_c in a distance equal to x_c, whereas the asymptotic value is 11% of P_c. At a distance of $10x_c$ the pressure is still a factor of 1.8 larger than the asymptotic value. The relative extent of the decay region increases as $\bar{\chi}_\infty$ increases or $x_c \rightarrow 0$. This is to be expected, since $d\delta^*/dx|_c$ increases at $\bar{\chi}_\infty$ increases, so that α_T is also increased. The effect of $\bar{\chi}_\infty|_c$ on the decay process should increase as α_w decreases.

It is reasonable to assume that significant centrifugal effects within the boundary layer should be confined to a region which extends away from the corner a distance which is comparable to δ^*_c. Hence, it is of interest to determine the decay in pressure and the growth of the displacement thickness at a distance of say $2\delta^*|_c$ downstream from the corner as predicted by the present model. If the model is to be self-consistent, the pressure decay and the boundary-layer growth should be small at this point. Some values of these quantities are summarized in Table 1 for the conditions given in Fig. 4. These results suggest that the model is reasonably self consistent and can be considered as a good first approximation.

Displacement thickness and heat-transfer distributions downstream of a corner for the same conditions used to calculate the results presented in Fig. 4 are given in Figs. 5 and 6. Significant decreases in heat-transfer rates are observed as would be expected. There are also very large increases in displacement thickness. In Fig. 7 the displacement thickness and pressure distributions in the immediate neighborhood of the corner are plotted for $Z_c = 0.163$, $P_c = 1.25$, $M_\infty\ \alpha_w = 1.0$, and $M_\infty = 10$; this corresponds to a corner angle $\alpha_w = 5.73°$. This representation gives a better indication of the

Table 1 Boundary-layer growth and pressure decay immediately downstream of corner. Values of δ^*/δ^*_c and P/P_c are given at $(x - x_c) = 2\delta^*_c$, $M_\infty\ \alpha_w = 1.0$

No.	Z_c	P_c	$\delta^*_c/$	P/P_c	δ^*/δ^*_c
1	0.598^{-2}	2.92	0.127	0.792	1.340
2	0.190×10^{-1}	1.97	0.0838	0.813	1.290
3	0.670×10^{-1}	1.44	0.0490	0.830	1.240
4	0.163	1.25	0.0320	0.842	1.220
5	0.803	1.10	0.0143	0.845	1.210

solution in the immediate neighborhood of the corner than is given in Figs. 4 and 6; and it confirms the general conclusions that can be inferred from Table 1.

Recently Igra[22] obtained some experimental results in the UTIAS 4 in. × 7 in. hypervelocity shock tube which were relevant to the present problem. Interferograms of the flow of ionized argon over a 15° corner expansion at a value of $M_c = 2.2$ and $g_w = 0.01$ were obtained, from which the disturbance thickness could be determined. The boundary-layer thickness distribution in the neighborhood of the corner obtained in this way is plotted in Fig. 8. Although a direct comparison of these results with the theoretical values predicted by the calculations presented here does not seem reasonable in view of the low value of M_c for the experiment, the general behavior of the results does lend support to the model.

Although the model presented here is based on the assumption $g_w \ll 1$ it is appropriate to comment on its applicability when g_w is not small. Clearly, the fundamental condition is $d\delta^*/dp_e|_c < 0$, and if this holds, then the major features of the physical picture should apply, whatever the value g_w happens to be. Of course, if g_w is not small, the cold wall similarity method should be replaced by a more sophisticated boundary-layer treatment. Clearly as g_w increases, it is expected that M_{ecr} will also increase. However, it is to be noted that the limiting hypersonic self similar solutions which exist for $p_e \sim x^n$ can exhibit a supercritical behavior, since $\delta^* \sim x^{(1-n)/2}$. Since these solutions are valid for adiabatic walls, they suggest that for sufficiently high M_e even an adiabatic boundary layer can be made to go supercritical, so that the present model would apply.

The present theory may be contrasted with the work of Zakkay et al.[1,2] As noted in the introduction, their approach used a more realistic treatment of centrifugal effects, but ignored the downstream interaction process. Their approach gave for a 20° half-angle sharp cone-cylinder, and a value of $M_c = 4.25$ an increase in δ of a factor of about 2 immediately downstream of the corner. With this approach, the factor should increase with increase in M_c. The present method will give almost negligible change in δ immediately downstream of the corner, even for turning angles α_w as large as 20°. It seems likely that their method is more appropriate

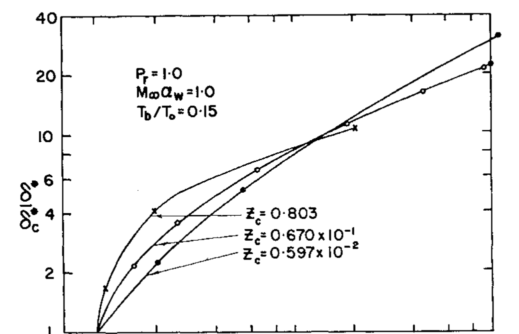

Fig. 6 Growth of the boundary layer downstream of the corner caused by the interaction of the boundary layer and the expansion wave.

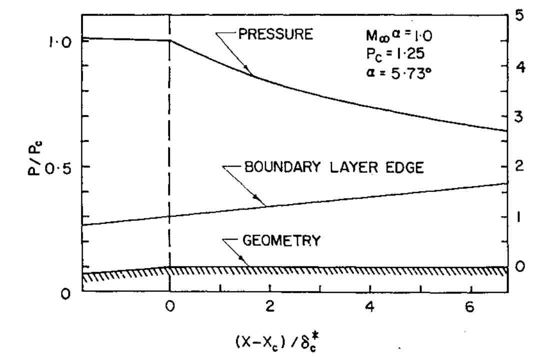

Fig. 7 Pressure and displacement thickness distribution in the immediate neighborhood of corner 2 for $M_\infty = 10$.

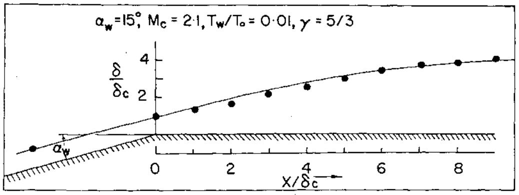

Fig. 8 Boundary-layer thickness distribution near a corner expansion as determined from an interferogram taken in the UTIAS 4 in. × 7 in. shock tube.

for low values of M_c and high values of α_w, whereas the present theory is restricted to large M_c and small α_w.

7. Conclusions

Although the present calculations are based on a very simple model of this problem, they do demonstrate the importance of the downstream interaction process for locally hypersonic flows. There is clearly a need for suitable experiments designed to establish the importance of this interaction. The crucial measurements which would establish the existence of this effect are pressure distributions on two-dimensional models. However, there is also a need for measurements of boundary-layer profiles near the corner and to examine the effect of g_w experimentally.

Further theoretical calculations are also needed. None of the existing work, including that presented here, predicts the expected peak in the heat-transfer rate at the corner itself.

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